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Evolution of perturbations on a weakly inhomogeneous background☆

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Abstract

The evolution of growing and decaying one-dimensional linear perturbations on a stationary, weakly inhomogeneous background is investigated studied. Attention is focused on the amplification of waves that arise from initial perturbations, localized in regions whose width is small compared with the inhomogeneity scale. A relation between the Hamiltonian formalism (with a complex dispersion equation) and the saddle-point method is established for an asymptotic representation of the integral that expresses perturbations in terms of the initial data. Model examples of the evolution of perturbations are examined.

1. Investigation of perturbations using the Hamiltonian formalism

One-dimensional linear unsteady perturbations are examined on a weakly inhomogeneous stationary background, i.e., on a background that depends on the ratio of the x coordinate to the inhomgeneity scale L, where the asymptotic properties of the perturbations are considered large values of L. At the same time, it is assumed that the characteristic times t are also large. In small intervals of x and t variations, the inhomogeneity of the background is insignificant, and perturbations of a general type can be regarded as consisting of elementary solutions, whose dependence on x and t is of the form $\exp\,[i(kx-\omega t)]$. The equations describing the perturbations specify a relation between the wave number k and the frequency ω, which is called a dispersion equation. In the case of a weakly inhomogeneous background, this provides a basis for searching for a solution in the form.[1]

$$\exp[i\Theta(x,t)],\quad \partial\Theta/\partial x = k,\quad \partial\Theta/\partial t = -\omega,\quad \omega = \Omega(k,x) \tag{1.1}$$

The last relation in (1.1) is a dispersion equation solved for ω. In the case under consideration, the dispersion equation includes a slow dependence on x.

When the two preceding equalities in (1.1) are taken into account, this dispersion equation is a first-order equation in Θ, which is called the Hamilton–Jacobi equation

$$\partial\Theta/\partial t + \Omega(\partial\Theta/\partial x, x) = 0$$

Solutions of the Hamilton–Jacobi equation can be obtained by integrating the system of characteristic equations

$$dx/dt = \partial\Omega/\partial k,\quad dk/dt = -\partial\Omega/\partial x \tag{1.2}$$

$$d\Theta/dt = k\,dx/dt - \omega(x,k) \tag{1.3}$$

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The Eq. (1.2) are called Hamilton equations. The function $\Omega(k, x)$ is the Hamiltonian. Hamilton equations were originally intended for studying waves with a real dispersion equation, i.e., an equation in which k and ω are real.[1] The use of the Hamiltonian approach in the case of complex dispersion equations that describe decaying or growing perturbations was proposed in Ref. 2. The Hamiltonian method was proposed in Ref. 3 for studying the instability of weakly inhomogeneous states and finding the eigenfrequencies and eigenfunctions. The spectrum was investigated in problems with a complex Hamilton equation in Refs 4,5. A similar approach has also been used to analyse the growth of perturbations and instability on particle trajectories in an incompressible fluid.[6,7]

The use of the Hamiltonian formalism is of interest not only for concluding whether the process being considered is stable or unstable, but also for the possibility of tracing the evolution of perturbations for long, but finite time intervals. An asymptotic analogy of Green's function can then be found for complicate systems of equations when the initial data are concentrated in a region that is small compared with L.

A similar problem was previously considered in Ref. 8, where finding the solution in the general case was reduced to constructing series in increments of x and t, and the series obtained could be summed only in a few special cases. The Hamiltonian formalism has the advantage that it to find an asymptotic solution at once in a large region with respect to x and t in all cases.

If the dispersion equation is complex, the integral curves of the Hamilton equations start out from real values of x and, in general, pass through complex values of x. Suppose perturbations are specified in a vicinity of the initial point $x = x_0$, $t = 0$, and suppose information regarding the solution should be obtained at the observation point x, t. The problem can be formulated as follows: it is required to find a value (a complex value in the general case) of ω_0 such that the integral curve of Eq. (1.2) starts out from the point x_0, 0 and reaches the point x, t.[2] The function Θ, which enables the amplification factor $-\mathrm{Im}\,\Theta$ and its phase $\mathrm{Re}\,\Theta$ to be calculated by integrating Eq. (1.3) together with Eq. (1.2). We shall call the curve in the complex x plane, corresponding to the function $x(t)$ obtained as a solution of the Hamilton equations, a trajectory.

In the problem under consideration, we can take advantage of the fact that Hamilton's equations have a known integral: the Hamiltonian does not vary along the integral curves of Hamilton's equations. This enables as to replace one of Hamilton's equations by this integral

$$\Omega(k, x) = \omega = \text{const} \tag{1.4}$$

2. The relation between the Hamiltonian formalism and the saddle-point method

Let an initial perturbation be specified at $t = 0$ in a small vicinity of the point x_0. We assume that the vicinity is so small compared with the scale L that the perturbation may be represented in the form of the integral

$$\int_{-\infty}^{\infty} f(k_0)\exp(ik_0x)dk_0 \tag{2.1}$$

The integration is carried out over real k_0 (the solution can represent a sum of terms of the type indicated, and $f(k_0)$ can be a vector function).

If the times for which the perturbation cannot leave the vicinity of the point x_0 with a width much less than L are considered, the perturbation acquires a form in which the inhomogeneity of the background is not taken into account:

$$\int_{-\infty}^{\infty} f(k_0)\exp i[k_0(x - x_0) - \omega(k_0)t]dk_0, \quad \omega(k_0) = \Omega(k_0, x_0) \tag{2.2}$$

The last equality in (2.2) defines ω as a function that represents the dispersion equation at $x = x_0$.

The standard calculation of the asymptotic form of integral (2.2) for large values of t (which are, however, less than the values at which the perturbation propagates so far that the inhomogeneity begins to have an influence) reduces[9] to finding the saddle point[10] for the integral (2.2). This point is a stationary point of the complex exponent of the exponential function, and the absolute value of the exponential function at the saddle point specifies the amplification factor of the initial perturbation. The wave amplification extremum is thus found. However, only the stationary point of the exponent which is a saddle point for the integral (2.2), rather than any stationary point, gives an asymptotic. In the general case, stationary points lying aside the integration path, which are not related to the asymptotic behaviour of the integral, can exist.

When the behaviour of perturbations is investigated at still longer times, at which the inhomogeneity has an effect, the exponent of the exponential function in (2.2) should be replaced by $i\Theta$:

$$\int_{-\infty}^{\infty} f(k_0)\exp[i\Theta(k_0)]dk_0 \tag{2.3}$$

The function $\Theta(k_0)$ is defined according to Eq. (1.3):

$$\Theta(k_0) = \Theta(\omega(k_0)) = -\omega t + \int_{x_0}^{x} k(\omega, x)dx \tag{2.4}$$

The other arguments on which Θ depends, viz., x_0, x and t, are assumed here to be specified and were therefore not written down, and the function $\omega(k_0)$ is specified by the second equality in (2.2).

If we start out from the Hamiltonian formalism, the integration in (2.4) should be carried out over the integral curve of Hamilton's equations. However, the integration path of the analytic function $k(\omega, x)$ can be altered without changing the values of the integral. Note that if the integral in (2.4) is represented in terms of the average value of the integrand as $k_{\text{av}}(x - x_0)$, the exponent of the exponential function in integral (2.3) takes the form

$$i\Theta = ik_{\text{av}}(x_1 - x_0) - i\omega t$$

and differs from the exponent in integral (2.2) by the fact that k_0 is replaced here by the average value k_{av}, which depends on x_0, x, t and k_0 and differs from k_0 in the general case. For large L and t the value of Θ is large, and the main contribution to integral (2.3) is made by the saddle point, where the derivative $\partial\Theta/\partial k_0$ vanishes (i.e., it is a stationary point of the function Θ).

The stationary points of the function Θ can be sought in the following manner. After equating the derivative of the right-hand side of Eq. (2.4) with respect to k_0 to zero and multiplying the resultant equality by $dk_0/d\omega$, we obtain the equation

$$\int_{x_0}^{x} \frac{\partial k}{\partial \omega}dx - t = 0 \tag{2.5}$$

which specifies the value of ω (or k_0) at the stationary points of the function $\Theta(k_0)$ for fixed values of x_0, x and t. Assuming that ω and x_0 are constant and that x and t are variable, we differentiate equality (2.5) and obtain

$$-\frac{\partial k}{\partial \omega}dx + dt = 0, \quad \omega = \text{const}$$

These equalities are equivalent to Hamilton's Eq. (1.2).

Thus, the integral curve of Hamilton's equations coincides with the set of points x, each of which satisfies the extremum condition (2.5) for an assigned value of ω. Under specific conditions the inverse statement also holds: if the extremum condition (2.5) is satisfied for the point x_1, t_1, this point can be connected to the initial point x_0, 0 by the integral curve of Hamilton's equations. In fact, it follows from Eq. (2.5) that the trajectory obtained from Hamilton's equations can be defined in the complex x plane as the line on which

$$\operatorname{Im}\int_{x_0}^{x} \frac{\partial k}{\partial \omega}dx = 0, \quad \omega = \text{const} \tag{2.6}$$

According to Eq. (2.5), the real part of this integral will represent the time t.

If condition (2.5) holds, the initial point and the observation point lie on the same level line of the function, which represents the imaginary part of the indefinite integral of $\partial k/\partial\omega$ over x, and the level line of this function is the trajectory described by Hamilton's equation in the complex x plane. In order for a trajectory that connects the initial and final points to exist, these points must belong to the same branch of the level line (if there are several branches) or to different branches that merge at infinity (if the time needed to reach an infinitely distant point and return is finite.)

In the complex x plane, for an assigned frequency ω, the branching points of the integral

$$\int_{x_0}^{x} k(\omega, x)dx$$

which appears in expression (2.4) for Θ, are the points at which two or more branches of the multi valued analytic function $k(\omega, x)$ coincide if Θ is regarded as a function of the upper limit. In the WKB approximation, such points are called turning points.[11]

Consider the behaviour of trajectories obtained from Hamilton's equations in the complex x plane in a vicinity of a simple turning point, at which two branches of the function $k(\omega, x)$ for a specified value of ω coincide. In the vicinity of such a point, the following expansion holds

$$\Delta\omega = g(\Delta k)^2 + h\Delta x$$

where Δ denotes the deviations of the quantities from their values at the turning point, and g and h are constants, which are complex in the general case. When this expansion is used, it is not difficult to verify that the trajectory $\Delta x(t)$ obtained from Hamilton's equations has the form of a parabola in the vicinity of the turning point. When $\Delta\omega = 0$, the trajectory consists of a ray that emerges from the turning point and can be traversed twice in opposite directions. The value of Δk changes sign, and the trajectory passes from one sheet of the Riemann surface of the function $k(\omega, x)$ to another.

The quantity Θ, represented by Eq. (2.4) is a multi-valued function of ω (and, accordingly, of k_0). This must be taken into account when evolutions the integral in Eq. (2.4). All possible integration paths in the complex x plane should be considered, and different values of the integral will correspond to different routes for bypassing the turning points. In the general case, several trajectories obtained from Hamilton's equations and which pass from the initial point to an assigned observation point x, t for different values of k corresponding to different routes for bypassing the turning points, i.e., to different branches of the function $k(\omega, x)$, can exist. Physically, this corresponds to the fact that the perturbations can reflect from inhomogeneities of the background, which manifest themselves by the presence of the turning points. If such a situation exists, the amplification of perturbations corresponding to different paths of approach to the observation point for a complex dispersion equation may differ in the general case, and the largest among them can be selected, while the others are ignored.

Thus, there is a relation between the Hamiltonian formalism and the stationary points of the complex amplification factor $\Theta(k_0)$ defined by Eq. (2.4). In order for the value of Θ, which can be obtained, according to Eq. (1.3), by integrating the Hamilton's equations, to represent the asymptotic of the perturbations at long times, the corresponding stationary point of the function $\Theta(k_0)$ should be a saddle point for the integral (2.3) and should not lie aside path for integration over k_0. This set of conditions must be verified in particular cases. For example, the case considered below (Section 4) shows that the amplification factor Θ calculated from the integral curves of Hamilton's equations may fail to correspond to the asymptotic behaviour of the perturbations. On the other hand, in some cases it is helpful to use Hamilton's equations to search for the stationary points of the function $\Theta(k_0)$. By representing the solution in the form of the integral asymptotic, we can find not only the leading term of the asymptotic, but also the following terms of the expansion on an inhomogeneous background.

Note that the asymptotic value of integral (2.3) obtained by the saddle-point method, contains not only an exponential multiplier, which is specified by the value of Θ at the saddle point, but also another multiplier, which depends on the initial data and the value of $\partial^2\Theta/\partial k_0^2$ at the saddle point. In the examples considered below this multiplier is not taken into account, and only the amplification of the waves associated with Θ is studied because the exponential multiplier is more important for large L in the case of complex dispersion equations.

Note that Hamilton's equations written for the dispersion equation

$$\omega = \nu\Omega(\lambda k, \mu x)$$

(ν, λ and μ are constants) become identical to Hamilton's Eq. (1.2), which correspond to $\nu = \lambda = \mu = 1$, if the time t is replaced in these equations by $\tau = (\mu\lambda/\nu)t$. This allows of simultaneous examination of the behaviour of perturbations for families of dispersion equations that differ by the values of ν, λ and μ.

3. An example of the evolution of perturbations in systems with dissipation

Let us examine the behaviour of perturbations whose dispersion equation has the form

$$\omega = i(k - k^2) + kx \tag{3.1}$$

Such a dispersion equation corresponds, for example, to the behaviour of linear perturbations of a combustion front when the heat released due to combustion is small,[12] and the tangential component of the velocity in the unperturbed flow depends linearly on the x coordinate measured along the front. The second term on the right-hand side of Eq. (3.1) represents the influence of the drift of the perturbations along the x axis, whose velocity was set equal to x, on the frequency ω (the Doppler effect). This dispersion equation can serve as a model for describing instability in systems with dissipation (the instability for large k is suppressed by the dissipative term $-ik^2$).

Hamilton's equations corresponding to Eq. (3.1)

$$\frac{dk}{dt} = -k, \quad x = \frac{\omega}{k} - i + ik \tag{3.2}$$

are integrated explicitly. The last equality in (3.2) is a corollary of Eq. (3.1).

Taking advantage of these equations and the fact that $x = x_0$ when $t = 0$, and $x = x_1$ when $t = t_1$, we obtain

$$k = k_0 e^{-t}, \quad k_0 = \frac{e^{t_1} - 1}{e^{t_1} - e^{-t_1}} + i\frac{x_1 - x_0 e^{t_1}}{e^{t_1} - e^{-t_1}}, \quad x = \frac{\omega}{k_0}e^t - i + ik_0 e^{-t} \tag{3.3}$$

Thus, the trajectory $x(t)$ of interest is complex. When $t \neq 0$ it reaches a real value equal to x_1 only once when $t = t_1$. The quantity $-\mathrm{Im}\,\Theta^*$, which is the amplification factor of the perturbations at the point x_1, t_1, is also calculated explicitly:

$$\begin{gathered}-\Theta^* = \omega t_1 - \int_{x_0}^{x_1} k_0 e^{-t}\frac{dx}{dt}dt = -\frac{i}{2}(1 - e^{-2t_1})k_0^2, \quad -\mathrm{Im}\Theta^* = \frac{1}{2}(1 - e^{-2t_1})\mathrm{Re}k_0^2 2 \\ -\lim_{t\to\infty}\mathrm{Im}\Theta^* = \frac{1}{2}(1 - (x_1 e^{-t_1} - x_0)^2)\end{gathered} \tag{3.4}$$

In evaluating the integral, the derivative dx/dt was obtained using Eq. (3.3). Thus, the amplification factor is bounded everywhere, and on the curves $x_1 = (x_0 + q)e^{t_1} - q$ (q = const), as $t_1 \to \infty$ it takes the value

$$-\mathrm{Im}\Theta^* = \frac{1}{2}(1 - q^2)$$

The region where this limiting amplification factor is positive lies between the curves

$$x_1(t_1) = (x_0 - 1)e^{t_1} + 1 \quad \text{и} \quad x_2(t_1) = (x_0 + 1)e^{t_1} - 1$$

It reaches a maximum, equal to 1/2, on the curve $x_1(t_1) = (x_0 - 1)e^{t_1} + 1$.

Expression (3.4) for Θ, which was calculated using Hamilton's equations and is denoted by Θ^*, depends only on the coordinates of the initial and final points x_0, 0 and x_1, t_1, since the value of k_0 specified in Eq. (3.3), which ensures the relation between the initial and final points of the integral curve of Hamilton's equations, should be substituted into expression (3.4). According to Section 2, for assigned initial and final points, Θ^* represents a stationary value of the function $\Theta(k_0)$, obtained from (2.4) for the same initial and final points and an arbitrary value of k_0.

If k_0 tends to $\pm\infty$ along the imaginary axis in the complex k_0 plane, then Im $\omega(k_0) \to \infty$ as $|k_0|^2$. The function $\Theta(k_0)$ behaves in the same way, since the integral in expression (2.4) for fixed values of x_0, x_1 and t_1 is of the order of $\sqrt{\omega}$. Since the stationary point of the function $\Theta(k_0)$, which is represented by expression (3.3) for k_0, is unique (this can be verified by a direct calculation), this point is also a saddle point for integral (2.3). Therefore, for large values of the characteristic scale L of the problem, expression (3.4) for $-\mathrm{Im}\Theta^*$ and expression (3.3), which gives the function $k(t_1)$, describe the asymptotic behaviour of the amplitude and the wave number.

Fig. 1 shows the level lines of the amplification factor $-\mathrm{Im}\,\Theta^*$ in the x, t plane. The values of this quantity are indicated by the numbers on each of the lines.

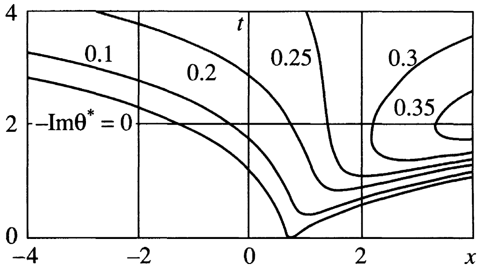

Fig. 1.

Note that in the homogeneous case (in which $x=x_0=\text{const}$ in (3.1)), growth of the perturbations occurs in the x, t plane on the straight lines $x=x_0+vt$ ($v=\text{const}$) at $x_0-\sqrt{2}<v<x_0+\sqrt{2}$ where the amplification factor increases linearly with time. This sector can be seen in Fig. 1 if we confine ourselves to the region of small variations of x and t. However, as the time increases further, growth of the perturbations becomes bounded everywhere in the inhomogeneous case, and the region of non-decaying perturbations broadens exponentially.

For better visualization of the results obtained, it is convenient to examine the first of Hamilton's Eq. (1.2). If the variable $y=1-ix$ is introduced, this equation can be brought to the form

$$\frac{dy}{dt} = \pm\sqrt{(y-c)(y+c)}, \quad k = \frac{y \mp \sqrt{y^2+4i\omega}}{2}, \quad c = \sqrt{-4i\omega} \tag{3.5}$$

It can be seen that the argument dy/dt is a half-sum of the arguments of the multipliers of the radicand, i.e., the integral curves of Eq. (3.5) are hyperbolae with foci at $\pm c$. These hyperbolae can run in either direction, depending on the choice of the sign in equalities (3.5). The points $\pm c$ in the complex y plane (or $-i\pm ic$ in the complex x plane) correspond to branch points of the function $k(\omega, y)$ (or $k(\omega, x)$). Fig. 2 shows a qualitative representation of such a trajectory in the complex x plane.

It can be seen from Fig. 2 that when the point $-i+ic$ lies in the lower half-plane of x, in order for the integral curve to return to the real x axis, it must pass below this branch point, i.e., it must return to the real axis with a value of k that differs from the value which it obtains by continuous running of the variable $k(\omega, x)$ along the real x axis.

If $|\omega|$ and, therefore, $|c|$ decrease, the turning points come together, and dx/dt and $d\Theta/dt$ become small in their vicinity. In addition, t_1 increases, and $\Theta(t_1)$ tends to a finite limit as $t_1\to\infty$.

Since there are two turning points, an eigenfunction can be constructed. To accomplish this, the integral between the branch points should take real positive values that are multiples of $\pi/2$:[11,13–15]

$$\int_{-i-ic}^{-i+ic} (k_2-k_1)dx = \pi\left(n+\frac{1}{2}\right), \quad n = 0, 1, 2\ldots$$

The eigenfrequencies $\omega=\omega_n$ are determined for each n from this equation. The integral on the left-hand side of the equality can be transformed in the following manner

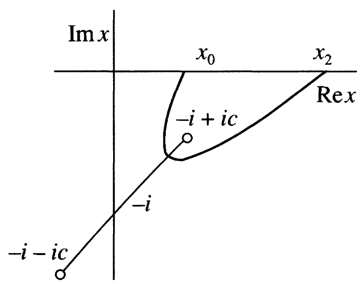

Fig. 2.

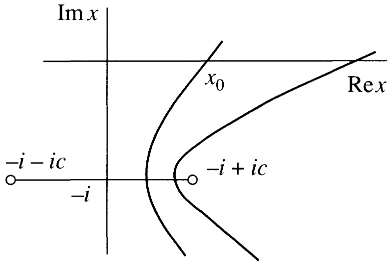

Fig. 3.

$$\int_{-\sqrt{4i\omega}}^{\sqrt{4i\omega}} \sqrt{4i\omega-\xi^2}d\xi, \quad \xi = x+i$$

Hence it can be seen that $\omega_n = -ir_n$, $r_n > 0$. The integration is performed over the real values of ξ between the turning points (Fig. 3).

Thus, all the eigenmodes of the perturbations decay exponentially with time. However, when the scale L is increased (this can be achieved by varying λ in the transformations described at the end of Section 2), the rate of decay of the first mode tends to zero. This is consistent with the fact that, according to the Hamiltonian formalism, the growth of perturbations, as was shown, tends to a finite limit as $t_1 \to \infty$, and the growth rate of non-stationary perturbations also tends to zero as $t_1 \to \infty$. However, it is interesting that the trajectories determined from Hamilton's equations for $\omega = -ir$, $r > 0$ move away from the real x axis and never return to it. They intersect the segment joining the branch points at a right angle, and then Im x tends to $-\infty$ on them. A qualitative representation of the behaviour of such trajectories is shown in Fig. 3.

4. An example of non-dissipative instability

We will consider the behaviour of perturbations for which the dispersion equation has the form

$$\omega = ak \pm \sqrt{k^2 - x} \tag{4.1}$$

When $x < 0$ this equation relates real values of ω to real values of k, i.e., the conditions of local stability holds, and for each $x > 0$ there is a range of values of k for which $\mathrm{Im}(\omega) \neq 0$. The most interesting case is $0 < a < 1$, in which the condition of local absolute instability holds when $x > 0$. The differential equation

$$\frac{\partial^2 u}{\partial t^2} - 2a\frac{\partial^2 u}{\partial t \partial x} - (1-a^2)\frac{\partial^2 u}{\partial x^2} = -xu$$

can be reconciled with dispersion Eq. (4.1). It is a hyperbolic differential equation with the characteristic velocities $1+a$ and $-1+a$.

Eq. (4.1) is a model equation that describes the behaviour of envelope waves for systems of equations without dissipation. A relation of type (4.1) that has a positive constant instead of x and contains the non-linear terms that were omitted in (4.1) was used in the dispersion equation in Ref. 16 to describe envelope waves on the surface of a tangential discontinuity between ideal fluids when surface tension and gravity are taken into account.

As was done previously, we will assume that the initial data at $t = 0$ are non-zero in a small vicinity (compared with the inhomogeneity scale) of the point x_0 and that they are expanded in a Fourier integral with respect to the wave number k_0 that satisfies the equation relating ω and k_0:

$$\omega = ak_0 \pm \sqrt{k_0^2 - x_0}, \quad x_0 = \mathrm{const} \tag{4.2}$$

The wave number $k(\omega, x)$ that satisfies Eq. (4.1) may be regarded as a function of k_0 and x: $k = k(k_0, x)$. We introduce the notation

$$\tilde{\omega}(k_0, x_0, x_1, t) = \omega - \frac{1}{t}\int_{x_0}^{x_1} k(\omega, x)dx \tag{4.3}$$

According to relation (2.4), the quantity $\tilde{\omega}$ thus defined is equal to $-\Theta/t$.

We will first consider the case of very small t and $x_1 - x_0$. Then dispersion Eq. (4.2) can be used, and the integral can be replaced by $k_0(x_1 - x_0)$. Here the quantity $\tilde{\omega}$ introduced above is the frequency in a system of coordinates that moves with a velocity v relative to the original system of coordinates:

$$\tilde{\omega}_0(k_0, v) = \omega(k_0) - vk_0, \quad v = \frac{x_1 - x_0}{t} \tag{4.4}$$

It is not difficult to see that for real k_0 expression (4.2) defines the real function $\omega(k_0)$, which corresponds to a hyperbola with centre at the origin of coordinates and asymptotes $\omega = (a+1)/k_0$ and $\omega = (-1+a)k_0$ in the plane of real ω and k_0. Its branches are located in the half-planes $k_0 > 0$ and $k_0 < 0$.

If we assume that k_0 is a pure imaginary quantity ($k_0 = i\eta$), then $\omega = i\beta$, and the real function $\beta(\eta)$ is also represented by as a hyperbola in the plane of real η and β with the same centre and asymptotes. One branch is located in the half-plane $\beta > 0$, and the other branch is located in the half-plane $\beta < 0$. The stationary points of the function $\tilde{\omega}(k_0)$ specify two values of k_0^*, which differ in sign and satisfy the equation

$$\partial\tilde{\omega}/\partial k_0 = 0 \tag{4.5}$$

and when

$$-1 + a < v < 1 + a \tag{4.6}$$

they lie on the imaginary axis of the complex k_0 plane and correspond to the plus and minus signs in Eq. (4.2). For values of v outside this interval, these values lie on the real k_0 axis.

When the asymptotic for $t \to \infty$ of the Fourier integral (2.2), which is initially taken over real k_0, is considered, the saddle point that specifies the asymptotic behaviour of the solution when inequality (4.6) holds is the pure imaginary point $k_0^* = i\eta*$, at which $\omega^* = i\beta^*$ with a value of β^* that lies on the upper branch ($\beta^* > 0$) of the hyperbola (which corresponds to the plus sign in Eq. (4.2)). At any value of v in interval (4.6), $\tilde{\omega}$ takes pure imaginary values, and the value of $\tilde{\omega}$ is positive at the stationary point, which is denoted by $\tilde{\beta}^*$. As v tends to the boundaries of interval (4.6), $\tilde{\beta}^* \to 0, \beta^* \to \infty|\eta^*| \to \infty$.

If v lies outside interval (4.6), the stationary point is located on the real k_0 axis, and the integration contour of in the first equality in (2.2) can be displaced to a region of as high as desired positive or negative values of $\operatorname{Im} k_0$, where $\operatorname{Im}\tilde{\omega}(k_0)$ appears to be negative and its absolute value is as high as desired. This means that for such values of v the solution on the straight lines $x = vt + \text{const}$ decreases with time more rapidly than any exponential function, and the stationary point lying on the real k_0 axis does not contribute to integral (2.2). In fact, the solution on these straight lines vanishes after a finite time, since the dispersion Eqs. (4.1) and (4.2) correspond to hyperbolic equations, and the solution in the x, t plane is non-zero only in the region bounded by the characteristics, whose velocity is equal to $-1 + a$ and $1 + a$.

Returning to the problem with an inhomogeneous background (4.1), we replace the integral in Eq. (4.3) with the expression $k_{av}(x_1 - x_0)$, where k_{av} is the average value of k on the integration path. In the cases considered below, in which $k(k_0, x)$ is a real or pure imaginary quantity, we can find x_{av} such that

$$\frac{1}{t}\int_{x_0}^{x_1} k(k_0, x)dx = vk_{av} = vk(k_0, x_{av}) = vk(\omega, x_{av}), \quad v = (x_1 - x_0)/t$$

Here the variable k_0 has been replaced by ω according to equality (4.2). As a result, k_{av} satisfies equality (4.1) with x replaced by x_{av}. Considering the frequency ω as a function of k_{av}, we write equality (4.3) in the form

$$\tilde{\omega} = \omega(k_{av}) - vk_{av} \tag{4.7}$$

To find the saddle point, we examine this relation at the pure imaginary values $k_{av} = i\eta_{av}$, where ω and $\tilde{\omega}$ take the pure imaginary values $\omega = i\beta$ and $\tilde{\omega} = i\tilde{\beta}$. We henceforth confine ourselves to solely the positive branch of the function $\beta(\eta)$, since only this branch provides growing perturbations. Then, from Eq. (4.7) and the foregoing investigation of the homogeneous case, it follows for all v in interval (4.6) that the inequality $\mathrm{Im}\,\tilde{\omega} = \tilde{\beta} > 0$ holds and that there is a single value $\eta_{av} = \eta^*$ for which $d\tilde{\beta}/d\eta_{av} = 0$. Since as $d\eta_{av}/d\eta_0 \neq 0$ (this follows from the inequality $d\eta_{av}/d\eta_0 > 0$ when $x = \text{const}$), it follows from the latter equality that there is a single stationary point with a minimum value of Im $\tilde{\beta}$ equal to $\tilde{\beta}* > 0$ on the imaginary axis of the complex k_0 plane. The fact that it is the only stationary point on the imaginary axis of k_0 can also be verified by direct calculation.

In the k_0 plane this point is a saddle point for the Fourier integral that specifies the evolution of the perturbations. Since at this point $\tilde{\beta} = \tilde{\beta}* > 0$, the amplitudes of the perturbations increase when the final point x_1, t_1 moves along the straight line $x = x_0 + vt$ for values of v that satisfy inequalities (4.6). The behaviour of the perturbations will be examined in greater detail below using Hamilton's equations. If the value of v passes over one of the boundaries of interval (4.6), then, as follows from equality (4.7) and dispersion Eq. (4.1), k_{av}^* and ω^* at first go to infinity along the imaginary axis and then come to the real axis. The same behaviour is exhibited by k_0^*. If the value of v lies outside interval (4.6), the contour of integration can always be displaced in the complex k_0 plane so that the integrand in integral (2.3), which represents a perturbation, will decrease (as t increases) more rapidly than any exponential function. As was previously noted, for values of v outside interval (4.6) a perturbation on any straight line $x = vt + \text{const}$ vanishes after a finite time.

We will now consider the evolution of perturbations using Hamilton's equations that are equivalent to the following equalities:

$$\frac{dk}{dt} = \frac{1}{2(\omega - ak)}, \quad x = (1 - a^2)\left(k + \frac{a\omega}{1 - a^2}\right)^2 - \frac{\omega^2}{1 - a^2} \tag{4.8}$$

The first equality is a Hamilton's equation with a right-hand side that was transformed taking the dispersion equation into account. The second equality in (4.8) is the dispersion Eq. (4.1), solved for x.

To study growing perturbations, we set $k = i\eta$ and $\omega = i\beta$, and integrate the first equality in (4.8). We obtain

$$t = a\eta^2 - 2\beta\eta + t_0, \quad x = -(1 - a^2)\eta^2 - 2a\beta\eta + \beta^2 \tag{4.9}$$

The assignment of β and t_0 specifies the real trajectory $x(t)$, which represents the solution of Hamilton's equations. A qualitative representation of the corresponding dependence for $\beta = 1$ and $t_0 = 0$ is shown in Fig. 4 in the region $x > 0$.

The velocity dx/dt varies monotonically from $1 + a$ at $t = t_1 = -(2 + a)/(1 + a)^2$ to $-1 + a$ at $t = t_2 = (2 - a)/(1 - a)^2$ along the trajectory plotted in Fig. 4. The maximum value $x = x_m = 1/(1 - a^2)$ is reached at $t = t_3 = a/(1 - a^2)^2$.

Note that Hamilton's equations and the equalities that follow from them in (4.9) remain valid after the transformation

$$k \to \chi k, \quad \omega \to \chi\omega, \quad x \to \chi^2 x, \quad t \to \chi^2 t, \quad t_0 \to \chi^2 t_0 \tag{4.10}$$

with an arbitrary positive χ. This transformation is a special case of the transformations considered at the end of Section 2. Transformation (4.10) leaves the velocity dx/dt unchanged. If x_0, x_1 and the time interval t are specified, this

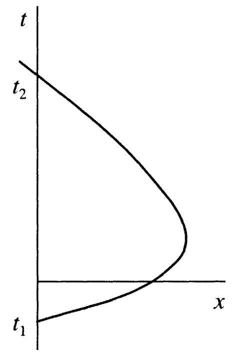

Fig. 4.

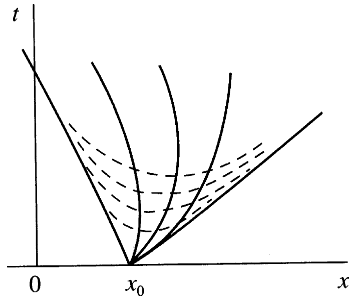

Fig. 5.

transformation, under which x and t are multiplied by χ^2, results in multiplication of the two terms in equality (4.3), which represent $\tilde{\omega}$, by χ. It is also obvious that the trajectories $x(t)$ allow of an arbitrary time shift.

The above transformations can be used to represent qualitatively the behaviour of the trajectories $x(t)$ emerging from the point $x = x_0$ in the x, t plane. Let the initial perturbation be specified at the point x_0. We shall examine the set of trajectories corresponding to all possible χ for which a suitable time shift can be chosen so that the trajectory passes through the point $x = x_0$, $t = 0$. A bundle of trajectories (represented by the solid curves in Fig. 5), whose slopes at the initial point are specified by inequalities (4.6), is obtained. On each trajectory the value $\omega = i\beta$ is maintained, while the value of $\eta = \mathrm{Im}\, k$ decreases with time according to the first equalities in (4.8) and (4.9). The quantity $\mathrm{Im}\, \tilde{\omega}*$ varies along the trajectory, but remains positive. Thus, in the case under consideration, in the x, t plane there is a region enclosed between the straight lines $x = x_0 + (1 - a)t$ and $x = x_0 - (1 - a)t$, in which perturbations grow with time without limit. As was shown above, outside this sector, there are no perturbations for large values of t.

Each of the trajectories, despite its drift with a positive velocity a, ultimately is forced out to the left from the instability region. This behaviour, which seems paradoxical at first, has a simple explanation. According to equalities (4.9), the value of η decreases, and the group velocity $\mathrm{d}\beta/\mathrm{d}\eta$ also decreases and tends to the value $-1 + a$, which is negative in the case under consideration.

Note that for real values of ω and k, the velocity corresponding to the integral curves of Hamilton's equations

$$dx/dt = \partial\omega/\partial k$$

has a real value that lies outside interval (4.6). As has been stated, perturbations cannot propagate with such a velocity. The stationary point of the function $\omega(k_0)$ lying on the real k_0 axis does not influence the asymptotic behaviour of the integral (as well as in the one-dimensional case).

In the case under consideration, the quantity Θ, defined by Eq. (2.4), is the pure imaginary quantity $i\Psi$, and the amplification factor is $-\Psi$. Using the equalities in (1.1), we find that the lines $\Psi = \mathrm{const}$ in the x, t plane are described by the equation

$$dx/dt = \omega/k$$

where ω and k are the frequency and wave number determined as the solution of Hamilton's equation.

The equality which follows from the particular form of the dispersion Eq. (4.1)

$$\omega/k = a + (\partial\omega/\partial k - a)$$

enables us to find the slope of the lines $\Psi = \mathrm{const}$ in the x, t plane if $\partial\omega/\partial k$ is known. This quantity is defined by the slope of the trajectories plotted in Fig. 5. Using it, we can construct a qualitative representation of the behaviour of the lines $\Psi = \mathrm{const}$ in the x, t plane. These are the dashed curves in Fig. 5. The amplification factor $-\Psi$ vanishes on the straight lines $x - x_0 = (1 - a)t$ and $x - x_0 = (-1 + a)t$. These straight lines are asymptotes for the level lines $\Psi = \mathrm{const}$. Note that it follows from equalities (4.9) for $x = x_0$ at $t = 0$ that ω and k tend to infinity as $\sqrt{t}$ at large values of t on the straight lines $x - x_0 = vt$, $-1 + a < v < 1 + a$, in agreement with transformation (4.10). Therefore, according to equalities (2.4), the amplification factor $-\Psi$ increases as $t^{3/2}$ in the x/t plane in the sector between the straight lines $x_1 = x_0 - (1 - a)t$ and $x_2 = x_0 + (1 + a)t$. This distinguishes the case under consideration from the case of a homogeneous background, where the perturbations grow in the same sector, but the amplification factor increases in proportion to t.

As is seen from Fig. 5, at long times all the perturbations that remain in the region $x>0$ develop from a narrow bundle of trajectories that emerge from the initial point with velocities close to $1+a$.

One more noteworthy feature of the example considered is the fact that perturbation growth occurs when there are no eigenfunctions. In fact, it follows from dispersion Eq. (4.1) that

$$k = [-a\omega \pm \sqrt{\omega^2 + x(1-a^2)}]/(1-a^2)$$

In this case there is a single turning point $x=\omega^2(1-a^2)^{-1}$. This makes it impossible to construct an eigenfunction using the WKB method, which is appropriate for the case under consideration.[9,11–13]

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References

1. Whitham GB. *Linear and Nonlinear Waves*. New York: Wiley; 1974.
2. Maslov VP. *Operator Methods*. Moscow: Nauka; 1973.
3. Iordanskii SV. Stability of inhomogeneous states and functional integrals. *Zh Eksp Teor Fiz* 1988;**94**(7):180–9.
4. Gal'tsev SV, Shafarevich AI. Quantized Riemann surfaces and quasiclassical spectral series for a non-self-conjugate Schrödinger operator with periodic coefficients. *Teor Mat Fiz* 2006;**148**(2):206–26.
5. Gal'tsev SV, Shafarevich AI. The Spectrum and pseudospectrum of a non-self-conjugate Schrödinger operator with periodic coefficients. *Mat Zametki* 2006;**80**(3):356–66.
6. Friedlander S, Vishik MM. Instability criteria for steady flows of perfect fluid. *Chaos* 1992;**2**(3):455–60.
7. Dobrokhotov SYu, Shafarevich AI. Some asymptotic solutions of the linearized Navier–Stokes equations. *Mat Zametki* 1993;**53**(1):25–35.
8. Hunt RE, Crighton DG. Instability of flows in spatially developing media. *Proc Roy Soc London Ser A* 1991;**435**(1893):109–28.
9. Lifshitz Ye M, Pitayevskii LP. *Physical Kinetics*. Moscow: Nauka; 1979.
10. Fedoryuk MV. *The Saddle-Point Method*. Moscow: Nauka; 1977.
11. Heading J. *An Introduction to Phase Integral Methods*. London, New York: Wiley; 1965.
12. Sivashinsky GI. Nonlinear analysis of hydrodynamic instability in laminar flames. I. Derivation of basic equations. *Acta Astronaut* 1977;**4**(11/12):1177–206.
13. Rukhadze AA, Silin VP. The geometrical optics method in the electrodynamics of an inhomogeneous plasma. *Usp Fiz Nauk* 1964;**82**(3):499–535.
14. Zaslavskii GM, Meitlis VP, Filonenko NN. *Interaction of Waves in Inhomogeneous Media*. Novosibirsk: Nauka; 1982.
15. Monkewitz PA, Huerre P, Chomaz JM. Global linear stability analysis of weakly non-parallel shear flows. *J Fluid Mech* 1993;**251**:1–20.
16. Weissman MA. Nonlinear wave packets in the Kelvin–Helmholtz instability. *Philos Trans Roy Soc London* 1979;**290**(1377):639–85.

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